

The Theory of Electrodynamics in a Linear Dielectric

Michael E. Crenshaw

*US Army Aviation and Missile Research, Development,
and Engineering Center, Redstone Arsenal, AL 35898, USA*

(Dated: March 7, 2013)

We adopt the continuum limit of a linear, isotropic, homogeneous, transparent, dispersion-negligible dielectric of refractive index n and examine the consequences of the effective speed of light in a stationary dielectric, c/n , for D'Alembert's principle and the Lagrange equations. The principles of dynamics in the dielectric-filled space are then applied to the electromagnetic Lagrangian and we derive the Hamiltonian and Hamilton's equations of motion for the fields and then macroscopic Maxwell equations. The resulting theory is mathematically equivalent to the familiar Maxwell-Heaviside formulation of continuum electrodynamics, but differs in the way that the material contribution is treated. A direct derivation of the symmetric total energy-momentum tensor from the field tensor for a dielectric is used to demonstrate the utility of the theory by resolving the Abraham-Minkowski electromagnetic momentum controversy.

I. INTRODUCTION

The dynamical equations of electromagnetism are known collectively as the Maxwell equations. In a material that responds linearly to electric and magnetic fields, the macroscopic Maxwell-Heaviside equations take the form

$$\nabla \times \mathbf{H} - \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t} = 0 \quad (1.1a)$$

$$\nabla \times \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} = 0 \quad (1.1b)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (1.1c)$$

$$\nabla \cdot \mathbf{D} = 0 \quad (1.1d)$$

in Heaviside-Lorentz units. Here, \mathbf{E} is the electric field, \mathbf{B} is the magnetic field, $\mathbf{D} = \epsilon \mathbf{E}$ is the displacement field, $\mathbf{H} = \mathbf{B}/\mu$ is the auxiliary magnetic field, ϵ is the electric permittivity, μ is the magnetic permeability, and c is the vacuum speed of light.

In this article, we derive equations of motion for macroscopic fields from a Hamiltonian treatment of electrodynamics in a linear dielectric. We adopt the continuum limit of a stationary linear dielectric of refractive index n and examine the consequences of the effective light speed c/n on D'Alembert's principle and the Lagrange equations. We derive the Hamiltonian and Hamilton's equations of motion for electromagnetic fields in a dielectric from the Lagrangian density. From Hamilton's equations, we derive macroscopic equations of motion

$$\nabla \times \mathbf{B} + \frac{n}{c} \frac{\partial \mathbf{\Pi}}{\partial t} = 0 \quad (1.2a)$$

$$\nabla \times \mathbf{\Pi} - \frac{n}{c} \frac{\partial \mathbf{B}}{\partial t} = \frac{\nabla n}{n} \times \mathbf{\Pi} \quad (1.2b)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (1.2c)$$

$$\nabla \cdot \mathbf{\Pi} = \frac{n}{c} \nabla \cdot \frac{\partial \mathbf{A}}{\partial t} + \frac{\nabla n}{n} \cdot \mathbf{\Pi} \quad (1.2d)$$

for the electric and magnetic fields in a stationary dielectric medium, where $\mathbf{B} = \nabla \times \mathbf{A}$ is the magnetic field and $\mathbf{\Pi} = (n/c) \partial \mathbf{A} / \partial t$ is the conjugate momentum field. With the exception of Poisson's Law, Eq. (1.2d), the equations of motion for the macroscopic fields, Eqs. (1.2), are mathematically equivalent to the Maxwell-Heaviside equations, Eqs. (1.1), and Poisson's Law conforms in the limit of static fields or the Coulomb gauge. As the success of a new theoretical construct is often gauged by its ability to resolve previously intractable problems, we use the variant form of Maxwell's equations, Eqs. (1.2), to derive the total electromagnetic energy-momentum tensor for the field in a dielectric, a result that has been sought for over a century. Thus, the Abraham-Minkowski momentum controversy [1, 2] is resolved in favor of a traceless symmetric total energy-momentum tensor that is based on the Gordon [3] form $\mathbf{g}_G = (\mathbf{B} \times \mathbf{\Pi})/c$ of the electromagnetic momentum density [4].

II. PARTICLE DYNAMICS IN A DIELECTRIC FILLED SPACE

In the continuum limit, a distribution of particles is regarded as a continuous medium and a property of the particles can be represented by a property density that is a continuous function of the spatial and temporal coordinates. In continuum electrodynamics, a linear medium is a region of space in which light travels from a source to an observer at a speed of c/n . We consider space to be entirely filled with a linear, isotropic, homogeneous, transparent, dielectric operating in a regime in which dispersion, electrostriction, and magnetostriction are negligible and, for convenience, we apply the term simple linear dielectric to this medium. Although dielectrics in the real world are much more complicated than this model of

a simple linear dielectric, theoretical models of physical phenomena are often based on reducing the complexity of the real world. Including too many details can hinder our understanding, especially if the theory becomes material specific, is based on phenomenological descriptions of effects, or becomes mathematically intractable.

In the rest frame of the simple linear dielectric medium, the constant refractive index n is the only property of a dielectric that is significant to the current problem. Here, we derive the dynamics of discrete particles as though the particles travel, without impediment, through a region of space that has the property that the speed of light is c/n . The dynamics of discrete particles is then generalized in the usual manner to derive the dynamics of electromagnetic fields in a dielectric.

Let the rest frame of the dielectric be $S(t, x, y, z)$ with orthogonal axes x , y , and z . Then position vectors in S are denoted by $\mathbf{x} = (x, y, z)$. If a light pulse is emitted from the origin at time $t = 0$, then

$$x^2 + y^2 + z^2 - \left(\frac{c}{n}t\right)^2 = 0 \quad (2.1)$$

describes wavefronts in the S system. Writing time as a spatial coordinate ct/n , the four-vector $(ct/n, \mathbf{x}) = (ct/n, x, y, z)$ represents the position of a point as a matter of geometry [5]. Then, we need to adapt the usual theory of dynamics of particles in the vacuum [6, 7] to this coordinate system.

For a system of particles, the transformation of the position vector \mathbf{x}_i of the i^{th} particle to J independent generalized coordinates is

$$\mathbf{x}_i = \mathbf{x}_i(\tau; q_1, q_2, \dots, q_J), \quad (2.2)$$

where $\tau = t/n$. Applying the chain rule, we obtain the virtual displacement

$$\delta \mathbf{x}_i = \sum_{j=1}^J \frac{\partial \mathbf{x}_i}{\partial q_j} \delta q_j \quad (2.3)$$

and the velocity

$$\mathbf{u}_i = \frac{d\mathbf{x}_i}{d\tau} = \sum_{j=1}^J \frac{\partial \mathbf{x}_i}{\partial q_j} \frac{dq_j}{d\tau} + \frac{\partial \mathbf{x}_i}{\partial \tau} \quad (2.4)$$

of the i^{th} particle in the new coordinate system. Substitution of

$$\frac{\partial \mathbf{u}_i}{\partial (dq_j/d\tau)} = \frac{\partial \mathbf{x}_i}{\partial q_j} \quad (2.5)$$

into the identity

$$\frac{d}{d\tau} \left(m \mathbf{u}_i \cdot \frac{\partial \mathbf{x}_i}{\partial q_j} \right) = m \frac{d\mathbf{u}_i}{d\tau} \cdot \frac{\partial \mathbf{x}_i}{\partial q_j} + m \mathbf{u}_i \cdot \frac{d}{d\tau} \left(\frac{\partial \mathbf{x}_i}{\partial q_j} \right) \quad (2.6)$$

yields

$$\frac{d\mathbf{p}_i}{d\tau} \cdot \frac{\partial \mathbf{x}_i}{\partial q_j} = \frac{d}{d\tau} \left(\frac{\partial}{\partial (dq_j/d\tau)} \frac{1}{2} m \mathbf{u}_i^2 \right) - \frac{\partial}{\partial q_j} \left(\frac{1}{2} m \mathbf{u}_i^2 \right). \quad (2.7)$$

For a system of particles in equilibrium, the virtual work of the applied forces \mathbf{f}_i vanishes and the virtual work on each particle vanishes leading to the principle of virtual work

$$\sum_i \mathbf{f}_i \cdot \delta \mathbf{x}_i = 0 \quad (2.8)$$

and D'Alembert's principle

$$\sum_i \left(\mathbf{f}_i - \frac{d\mathbf{p}_i}{d\tau} \right) \cdot \delta \mathbf{x}_i = 0. \quad (2.9)$$

Using Eqs. (2.3) and (2.7) and the kinetic energy of the i^{th} particle

$$T_i = \frac{1}{2} m \mathbf{u}_i^2, \quad (2.10)$$

we can write D'Alembert's principle, Eq. (2.9), as

$$\sum_j \left[\left(\frac{d}{d\tau} \left(\frac{\partial T}{\partial (dq_j/d\tau)} \right) - \frac{\partial T}{\partial q_j} \right) - Q_j \right] \delta q_j = 0 \quad (2.11)$$

in terms of the generalized forces

$$Q_j = \sum_i \mathbf{f}_i \cdot \frac{\partial \mathbf{x}_i}{\partial q_j}. \quad (2.12)$$

If the generalized forces come from a generalized scalar potential function V [6], then we can write Lagrange equations of motion

$$\frac{d}{d\tau} \left(\frac{\partial L}{\partial (dq_j/d\tau)} \right) - \frac{\partial L}{\partial q_j} = 0, \quad (2.13)$$

where $L = T - V$ is the Lagrangian. The canonical momentum is therefore

$$p_j = \frac{\partial L}{\partial (dq_j/d\tau)} \quad (2.14)$$

in a linear medium.

The field theory [7, 8] is based on a generalization of the discrete case in which the dynamics are derived from a Lagrangian density \mathcal{L} . The generalization of the Lagrange equation, Eq. (2.13), for fields in a linear medium is

$$\frac{d}{d\bar{x}_0} \frac{\partial \mathcal{L}}{\partial (\partial A_\nu / \partial \bar{x}_0)} = \frac{\partial \mathcal{L}}{\partial A_\nu} - \sum_i \partial_i \frac{\partial \mathcal{L}}{\partial (\partial_i A_\nu)}, \quad (2.15)$$

where $\bar{x}_0 = ct/n$ is the time-like coordinate in the material and x_1 , x_2 , and x_3 correspond to the respective x , y and z coordinates. We adopt the typical conventions that Roman indices run from one to three, Greek indices run from zero to three, and ∂_i represents the operator $\partial/\partial x_i$. The conjugate momentum field

$$\Pi_\nu = \frac{\partial \mathcal{L}}{\partial (\partial A_\nu / \partial \bar{x}_0)} \quad (2.16)$$

is used to construct the Hamiltonian density

$$\mathcal{H} = \sum_{\nu} \Pi_{\nu} \frac{\partial A_{\nu}}{\partial \bar{x}_0} - \mathcal{L} \quad (2.17)$$

from which Hamilton's equations of motion

$$\frac{\partial A_{\nu}}{\partial \bar{x}_0} = \frac{\partial \mathcal{H}}{\partial \Pi_{\nu}} \quad (2.18a)$$

$$\frac{\partial \Pi_{\nu}}{\partial \bar{x}_0} = -\frac{\partial \mathcal{H}}{\partial A_{\nu}} + \sum_i \partial_i \frac{\partial \mathcal{H}}{\partial (\partial_i A_{\nu})} \quad (2.18b)$$

are derived.

III. EQUATIONS OF MOTION FOR FIELDS IN A DIELECTRIC

We consider a stationary dielectric block illuminated at normal incidence from the vacuum by a plane quasimonochromatic electromagnetic pulse. The linear, isotropic, homogeneous, transparent, and dispersion-negligible dielectric is draped with a gradient-index antireflection coating. The dielectric is mechanically rigid and spatial variation of the refractive index is sufficiently smooth that reflection, and the associated radiation pressure, can be neglected. Then the refractive index is a smoothly varying real and time-independent function of position in a finite region of space.

We take the Lagrangian density of the electromagnetic field in the medium to be

$$\mathcal{L} = \frac{1}{2} \left(\left(\frac{\partial \mathbf{A}}{\partial \bar{x}_0} \right)^2 - (\nabla \times \mathbf{A})^2 \right) \quad (3.1)$$

in the absence of charges. Applying Eq. (2.16), the canonical momentum field

$$\mathbf{\Pi} = \frac{\partial \mathbf{A}}{\partial \bar{x}_0} \quad (3.2)$$

is substituted into Eq. (2.17) in order to construct the Hamiltonian density

$$\mathcal{H} = \frac{1}{2} (\mathbf{\Pi}^2 + (\nabla \times \mathbf{A})^2). \quad (3.3)$$

Hamilton's equations of motion in the dielectric

$$\frac{\partial \mathbf{A}}{\partial \bar{x}_0} = \mathbf{\Pi} \quad (3.4a)$$

$$\frac{\partial \mathbf{\Pi}}{\partial \bar{x}_0} = -\nabla \times \nabla \times \mathbf{A} \quad (3.4b)$$

are obtained from the Hamiltonian density, Eq. (3.3), using Eqs. (2.18). It is straightforward to reduce a second-order differential equation to first-order equations. To that end, we introduce a new field variable

$$\mathbf{B} = \nabla \times \mathbf{A}. \quad (3.5)$$

Substituting the definition of \mathbf{B} into Eq. (3.4b) produces the Maxwell–Ampère Law,

$$\nabla \times \mathbf{B} + \frac{\partial \mathbf{\Pi}}{\partial \bar{x}_0} = 0. \quad (3.6)$$

Thus, the second-order Hamilton's equation, Eq. (3.4b) has been written as two first-order equations, Eqs. (3.5) and (3.6), as promised. The curl of the other Hamilton's equation, Eq. (3.4a), is

$$\nabla \times \mathbf{\Pi} - \frac{\partial \mathbf{B}}{\partial \bar{x}_0} = \frac{\nabla n}{n} \times \mathbf{\Pi}, \quad (3.7)$$

known as Faraday's Law. The divergence of Eqs. (3.5) and (3.6) produces

$$\nabla \cdot \mathbf{B} = 0 \quad (3.8)$$

and Gauss's Law

$$\nabla \cdot \mathbf{\Pi} = \frac{n}{c} \nabla \cdot \frac{\partial \mathbf{A}}{\partial t} + \frac{\nabla n}{n} \cdot \mathbf{\Pi}. \quad (3.9)$$

The Maxwell–Heaviside equations, Eqs. (1.1), are based on the results of experimental studies of electricity and magnetism that were performed in the 18th and 19th centuries that were codified into physical laws. Bearing the names of Faraday, Ampère, and Gauss, not all of the laws of electricity and magnetism survived in their original form. Based on purely theoretical arguments of self-consistency, Maxwell [9] modified the Ampère law to include the effect of time-dependent electric fields. The changes to the Maxwell–Ampère and Faraday laws that are derived here are not nearly so profound. The two versions of these laws are mathematically equivalent, that is, they can be mutually transformed into each other using vector identities that commute a scalar function of position with the curl and divergence operators. However, it has been shown that these types of transformations alter the way various physical processes are expressed [10, 11]. There are also differences in the way some quantities are calculated. For example, derivation of the Fresnel relations requires conservation of energy in addition to Stokes's Law. In the next section, we show that the reformulation of the Maxwell equations allows us to resolve the century-old Abraham–Minkowski controversy [1, 2] for the electromagnetic momentum in a linear dielectric.

IV. FIELD AND ENERGY–MOMENTUM TENSORS

In the Maxwell–Heaviside formulation of classical continuum electrodynamics, there are two pairs of fields, $\{\mathbf{E}, \mathbf{B}\}$ and $\{\mathbf{D}, \mathbf{H}\}$, and two field tensors. Here, there is a single pair of fields $\{\mathbf{\Pi}, \mathbf{B}\}$ and a single field tensor. The field tensor,

$$F^{\alpha\beta} = \begin{bmatrix} 0 & \Pi_x & \Pi_y & \Pi_z \\ -\Pi_x & 0 & -B_z & B_y \\ -\Pi_y & B_z & 0 & -B_x \\ -\Pi_z & -B_y & B_x & 0 \end{bmatrix}, \quad (4.1)$$

is obtained in the usual way from

$$F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha \quad (4.2)$$

for homogeneous materials.

The reduction to a single field tensor and a single pair of fields results in a considerable simplification for certain derivations. For example, the energy–momentum tensor is defined in terms of the field tensor by [12]

$$T^{\alpha\beta} = -F^{\alpha\lambda}F_\lambda^\beta + \frac{1}{4}g^{\alpha\beta}F_{\lambda\nu}F^{\lambda\nu}, \quad (4.3)$$

such that

$$T^{\alpha\beta} = \begin{bmatrix} (\mathbf{\Pi}^2 + \mathbf{B}^2)/2 & (\mathbf{B} \times \mathbf{\Pi})_x & (\mathbf{B} \times \mathbf{\Pi})_y & (\mathbf{B} \times \mathbf{\Pi})_z \\ (\mathbf{B} \times \mathbf{\Pi})_x & W_{11} & W_{12} & W_{13} \\ (\mathbf{B} \times \mathbf{\Pi})_y & W_{21} & W_{22} & W_{23} \\ (\mathbf{B} \times \mathbf{\Pi})_z & W_{31} & W_{32} & W_{33} \end{bmatrix}, \quad (4.4)$$

where

$$W_{ij} = -\Pi_j\Pi_k - B_jB_k + \frac{1}{2}(\Pi^2 + B^2)\delta_{ij} \quad (4.5)$$

is the Maxwell stress tensor and $g^{\alpha\beta}$ is the diagonal metric tensor with non-zero elements $g^{00} = 1$ and $g^{ii} = -1$.

The form of the energy–momentum tensor has been debated for over a century [2]. The best known candidates are the 1908 Minkowski [15] tensor and the 1909 Abraham [16] tensor. Unlike previous candidates, $T^{\alpha\beta}$ satisfies necessary conditions for an energy–momentum tensor. The array is traceless and symmetric. Valid continuity equations [14]

$$\frac{\partial}{\partial \bar{x}_0} \left[\frac{1}{2}(\mathbf{\Pi}^2 + \mathbf{B}^2) \right] + \nabla \cdot (\mathbf{B} \times \mathbf{\Pi}) = 0 \quad (4.6a)$$

$$\frac{\partial}{\partial \bar{x}_0} (\mathbf{B} \times \mathbf{\Pi}) + \nabla \cdot \mathbf{W} = 0 \quad (4.6b)$$

are derived from the four-divergence $\bar{\partial}_\beta T^{\alpha\beta} = 0$ of $T^{\alpha\beta}$ [4], where

$$\bar{\partial}_\beta = \left(\frac{n}{c} \frac{\partial}{\partial t}, \partial_x, \partial_y, \partial_z \right) \quad (4.7)$$

is the four-divergence operator in the dielectric medium. Continuity equations and the energy–momentum tensor in an inhomogeneous dielectric are treated in Ref. [13]. Finally, integrating over all space σ , the quantities

$$U = \int_\sigma T^{00} dv, \quad (4.8a)$$

$$P^i = \frac{1}{c} \int_\sigma T^{i0} dv, \quad (4.8b)$$

are temporally invariant if the refractive index is a slowly varying function of position. The conservation of the total energy, Eq. (4.8a), and total momentum, Eq. (4.8b), can be viewed as uniqueness conditions for the elements of the total energy momentum tensor that represent the total energy density and total momentum density of an isolated system [14].

V. SUMMARY

We have recast classical continuum electrodynamics into a region of space in which the speed of light is c/n , instead of c . In principle, nothing is changed because the equations of motion can be transformed to the macroscopic Maxwell–Heaviside equations using vector identities: the two set of equations of motion are mathematically equivalent in dispersion-negligible transparent dielectrics. In practice, the new formulation introduces a new perspective on classical continuum electrodynamics. As a demonstration, we presented a one-line derivation of the total energy–momentum tensor, a result that has been sought for decades. This derivation provides affirmation for the total energy–momentum tensor that was recently obtained by construction from continuity equations [4] and from conservation properties [14].

-
- [1] B. A. Kemp, J. Appl. Phys. **109**, 111101 (2011).
 - [2] R. N. C. Pfeifer, T. A. Nieminen, N. R. Heckenberg, and H. Rubinsztein-Dunlop, Rev. Mod. Phys. **79**, 1197 (2007).
 - [3] J. P. Gordon, Phys. Rev. A **8**, 14 (1973).
 - [4] M. E. Crenshaw and T. B. Bahder, Opt. Commun. **284**, 2460 (2011).
 - [5] F. Ravndal, e-print arXiv:0804.4013v3.
 - [6] H. Goldstein, *Classical Mechanics*, 2nd ed. (Addison-Wesley, Reading, 1980).
 - [7] C. Cohen-Tannoudji, J. Dupont-Roc, and G. Grynberg,

- Photons and Atoms* (Wiley, New York, 1989).
- [8] M. Hillery and L. D. Mlodinow, Phys. Rev. A **30**, 1860 (1984).
- [9] J. C. Maxwell, Philosophical Magazine 161 (1861).
- [10] P. Kinsler, A. Favaro, and M. W. McCall, European J. Phys. **30**, 983 (2009).
- [11] W. Frias and A. I. Smolyakov, Phys. Rev. E **85**, 046606 (2012).
- [12] L. D. Landau and E. M. Lifshitz, *Electrodynamics of Continuous Media* (Addison-Wesley, Reading, 1960).
- [13] M. E. Crenshaw and T. B. Bahder, Opt. Commun. **285**,

- 5180 (2012).
- [14] M. E. Crenshaw, e-print arXiv:1302.6492.
- [15] H. Minkowski, Nachr. Ges. Wiss. Göttingen 53 (1908); Math. Ann. **68**, 472 (1910).
- [16] M. Abraham, Rend. Circ. Mat. Palermo **28**, 1 (1909); **30**, 33 (1910).